SEMSETRAL EXAMINATION
M. Math II YEAR, II SEMESTER, 2016-17
ERGODIC THEORY

Max. marks: 100 Time limit: 3hrs

Notation: $(\Omega, \mathcal{F}, P, T)$ denotes a dynamical system.

1. If T is ergodic but T^2 is not show that $P(E \cup T^{-1}E) = 1$ for some measurable set E with $P(E \cap T^{-1}E) = 0$.
Hint: take $E = A \backslash T^{-1}(A)$ for a suitable set A. [15]

2. Define $T : S^1 \times S^1 \to S^1 \times S^1$ by $T(u, v) = (uv^2, u^3v^7)$. Is T a homomorphism of the torus $S^1 \times S^1$?. If so, is it an automorphism? Does it preserve Haar measure on the torus? If so, is it ergodic? If so is it totaly ergodic? [20]

[You may use any result proved in class]

3. Let T be a bijection and T^{-1} be measurable. If T is ergodic with respect to another probability measure Q on $\mathcal{F}$ show that either $P \perp Q$ or $P = Q$. [15]

Hint: use Birkhoff's Ergodic Theorem.

4. Let $Tz = cz, z \in S^1$ where $c \in S^1$ is not a root of unity. Prove that $\frac{1}{n}\sum_{j=0}^{n-1} f(T^j z) \to \int f dP$ uniformly on S^1 for every $f \in C(S^1)$, P being the normalized Haar measure on S^1. [20]

Hint: $\int z^n dQ(z) = \int z^n dP(z)$ $\forall n \in \mathbb{Z}$ implies $P = Q$. **Do not assume Weyl's Theorem**.

5. For non-negative integer valued random variables X and Y define $H(X) = -\sum_{j=0}^{\infty} p\{X = j\} \log P\{X = j\}$, $H(X, Y) = -\sum_{j,k=0}^{\infty} p\{X = j, Y = k\} \log P\{X = j, Y = k\}$ and $H(X|Y) = -\sum_{j,k=0}^{\infty} p\{X = j, Y = k\} \log P\{X = j|Y = k\}$. Prove that $H(X, Y) = H(X|Y) + H(Y)$ and give an example to show that $H(X|Y)$ may not be equal to $H(Y|X)$. [10]

6. If X is a random variable possessing a density f define the entropy of X as $-\int_{-\infty}^{\infty} f(x) \log f(x) dx$. Give examples to show that $f \log f$ may not be integrable and, if it is, then $-\int_{-\infty}^{\infty} f(x) \log f(x) dx$ may be in $(0, \infty)$ or $(-\infty, 0)$. [20]